

Problem Set 4

Fall 2004

Issued: Wednesday, September 29, 2004

Due: Wednesday, October 6, 2004

Reading: For squeezed states:

- H.P. Yuen, “Two-photon coherent states of the radiation field,” *Phys. Rev. A* **13**, 2226-2243 (1976).
- L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995) Sects. 21.1–21.6.

For continuous-spectrum eigenkets:

- W.H. Louisell, *Quantum Statistical Properties of Radiation* (McGraw-Hill, New York, 1973) Sect. 1.10.

Problem 4.1

Here we shall show that the creation operator, \hat{a}^\dagger , does *not* have any non-zero eigenkets. Suppose that a non-zero ket $|\beta\rangle$ satisfies

$$\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle, \tag{1}$$

where β is a complex number. Use the completeness of the number kets to expand $|\beta\rangle$ as follows,

$$|\beta\rangle = \sum_{n=0}^{\infty} b_n |n\rangle,$$

where $b_n = \langle n|\beta\rangle$. Substitute this expansion into Eq. (1) and show that the only possible solution is $b_n = 0$ for all n , i.e., the creation operator has no non-zero eigenkets.

Problem 4.2

Here we shall work out some properties of the coherent states. Let \hat{a} and \hat{a}^\dagger be the annihilation and creation operators for the frequency- ω quantum harmonic oscillator discussed in class. Let $\{|\alpha\rangle : \alpha \in \mathcal{C}\}$ be the coherent states,

$$|\alpha\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \exp(-|\alpha|^2/2) |n\rangle,$$

where $\{|n\rangle : 0 \leq n < \infty\}$ are the number states and $\alpha \in \mathcal{C}$ is an arbitrary complex number.

- (a) Use the orthonormality of the number states, and the power series for the exponential function, to evaluate the inner product $\langle\alpha|\beta\rangle$ between two coherent states $|\alpha\rangle$ and $|\beta\rangle$. Are the coherent states normalized to unit length? Are coherent states with different eigenvalues orthogonal?
- (b) Use the completeness of the number states to show that the coherent states are overcomplete, i.e.,

$$\hat{I} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|,$$

where $d^2\alpha \equiv d\alpha_1 d\alpha_2$, with $\alpha_1 \equiv \text{Re}(\alpha)$ and $\alpha_2 \equiv \text{Im}(\alpha)$, and the integration region is the entire complex plane.

- (c) Use the result from (b) to show that,

$$\begin{aligned} \hat{a} &= \hat{a}\hat{I} = \int \frac{d^2\alpha}{\pi} \alpha |\alpha\rangle\langle\alpha|, \\ \hat{a}^\dagger &= \hat{I}\hat{a}^\dagger = \int \frac{d^2\alpha}{\pi} \alpha^* |\alpha\rangle\langle\alpha|, \\ \hat{a}\hat{a}^\dagger &= \hat{a}\hat{I}\hat{a}^\dagger = \int \frac{d^2\alpha}{\pi} |\alpha|^2 |\alpha\rangle\langle\alpha|, \\ \hat{a}^\dagger\hat{a} &= \hat{a}^\dagger\hat{I}\hat{a} - [\hat{a}, \hat{a}^\dagger] = \int \frac{d^2\alpha}{\pi} (|\alpha|^2 - 1) |\alpha\rangle\langle\alpha|. \end{aligned}$$

Problem 4.3

Here we shall develop a little commutator calculus that will be needed in the next problem. Let \hat{a} and \hat{a}^\dagger be the annihilation and creation operators, respectively, of a quantum harmonic oscillator, and let $\hat{a}_1 \equiv \text{Re}(\hat{a})$ and $\hat{a}_2 \equiv \text{Im}(\hat{a})$ be the associated quadrature operators, i.e., the normalized versions of position and momentum for a mechanical oscillator, or charge and flux for an LC oscillator.

- (a) Use $[\hat{a}_1, \hat{a}_2] = j/2$ to show that

$$[\hat{a}_1, \hat{a}_2^2] = j\hat{a}_2.$$

Assume that

$$[\hat{a}_1, \hat{a}_2^k] = jk\hat{a}_2^{k-1}/2, \quad \text{for } k > 2.$$

Show that

$$[\hat{a}_1, \hat{a}_2^{k+1}] = j(k+1)\hat{a}_2^k/2,$$

thus completing the induction proof that

$$[\hat{a}_1, \hat{a}_2^k] = jk\hat{a}_2^{k-1}/2, \quad \text{for } k = 1, 2, 3, \dots$$

By analogy with classical functions we *define* the following operator derivative,

$$\frac{d\hat{a}_2^k}{d\hat{a}_2} \equiv k\hat{a}_2^{k-1},$$

so that

$$[\hat{a}_1, \hat{a}_2^k] = (j/2) \frac{d\hat{a}_2^k}{d\hat{a}_2}, \quad \text{for } k > 2.$$

- (b) Follow a similar induction argument to that used in (a) to prove the commutation rule,

$$[\hat{a}_2, \hat{a}_1^k] = -jk\hat{a}_1^{k-1}/2 = -(j/2) \frac{d\hat{a}_1^k}{d\hat{a}_1}, \quad \text{for } k = 1, 2, 3, \dots,$$

where the last equality *defines* the operator derivative.

- (c) Suppose that $F(\alpha_1)$ and $G(\alpha_2)$ are functions of real variables α_1 and α_2 that have convergent Taylor's series,

$$F(\alpha_1) = \sum_{n=0}^{\infty} \frac{\alpha_1^n}{n!} \left. \frac{d^n F(\alpha_1)}{d\alpha_1^n} \right|_{\alpha_1=0}, \quad \text{for } -\infty < \alpha_1 < \infty,$$

$$G(\alpha_2) = \sum_{n=0}^{\infty} \frac{\alpha_2^n}{n!} \left. \frac{d^n G(\alpha_2)}{d\alpha_2^n} \right|_{\alpha_2=0}, \quad \text{for } -\infty < \alpha_2 < \infty.$$

Define the operators $F(\hat{a}_1)$ and $G(\hat{a}_2)$ by the operator-valued Taylor's series,

$$F(\hat{a}_1) = \sum_{n=0}^{\infty} \frac{\hat{a}_1^n}{n!} \left. \frac{d^n F(\alpha_1)}{d\alpha_1^n} \right|_{\alpha_1=0},$$

$$G(\hat{a}_2) = \sum_{n=0}^{\infty} \frac{\hat{a}_2^n}{n!} \left. \frac{d^n G(\alpha_2)}{d\alpha_2^n} \right|_{\alpha_2=0}.$$

Use the results of (a) and (b) to find the commutators $[\hat{a}_1, G(\hat{a}_2)]$ and $[\hat{a}_2, F(\hat{a}_1)]$.

Problem 4.4

Here we shall show that the eigenkets of a quadrature operator can be found from a translation operator applied to the zero-eigenvalue eigenket.

- (a) Assume that $|\alpha_1\rangle_1$ is an eigenket of the quadrature operator \hat{a}_1 with eigenvalue α_1 . Because \hat{a}_1 is Hermitian, α_1 is a real number. Define a translation operator,

$$\hat{A}_1(\xi) \equiv \exp(-2j\xi\hat{a}_2) = \sum_{n=0}^{\infty} \frac{(-2j\xi)^n}{n!} \hat{a}_2^n, \quad \text{for } -\infty < \xi < \infty.$$

Use

$$\hat{a}_1 \hat{A}_1(\xi) |\alpha_1\rangle_1 = \hat{A}_1(\xi) \hat{a}_1 |\alpha_1\rangle_1 + [\hat{a}_1, \hat{A}_1(\xi)] |\alpha_1\rangle_1,$$

and the results from Problem 4.3 to show that $\hat{A}_1(\xi) |\alpha_1\rangle_1$ is an eigenket of \hat{a}_1 with eigenvalue $\alpha_1 + \xi$, for any real number ξ .

(b) Let $|0\rangle_1$ be the \hat{a}_1 eigenket whose eigenvalue is zero. Show that

$$|\alpha_1\rangle_1 = \exp(-2j\alpha_1 \hat{a}_2) |0\rangle_1,$$

is an \hat{a}_1 eigenket with eigenvalue α_1 and that ${}_1\langle\alpha_1|\alpha_1\rangle_1 = {}_1\langle 0|0\rangle_1$.

(c) Assume that $|\alpha_2\rangle_2$ is an eigenket of the quadrature operator \hat{a}_2 with eigenvalue α_2 . Because \hat{a}_2 is Hermitian, α_2 is a real number. Define a translation operator,

$$\hat{A}_2(\xi) \equiv \exp(2j\xi \hat{a}_1) = \sum_{n=0}^{\infty} \frac{(2j\xi)^n}{n!} \hat{a}_1^n, \quad \text{for } -\infty < \xi < \infty.$$

Use

$$\hat{a}_2 \hat{A}_2(\xi) |\alpha_2\rangle_2 = \hat{A}_2(\xi) \hat{a}_2 |\alpha_2\rangle_2 + [\hat{a}_2, \hat{A}_2(\xi)] |\alpha_2\rangle_2,$$

and the results from Problem 4.3 to show that $\hat{A}_2(\xi) |\alpha_2\rangle_2$ is an eigenket of \hat{a}_2 with eigenvalue $\alpha_2 + \xi$, for any real number ξ .

(d) Let $|0\rangle_2$ be the \hat{a}_2 eigenket whose eigenvalue is zero. Show that

$$|\alpha_2\rangle_2 = \exp(2j\alpha_2 \hat{a}_1) |0\rangle_2,$$

is an \hat{a}_2 eigenket with eigenvalue α_2 and that ${}_2\langle\alpha_2|\alpha_2\rangle_2 = {}_2\langle 0|0\rangle_2$.

Problem 4.5

Here we shall continue our development of the quadrature-operator eigenkets. The results of Problem 4.4 show that these operators have continuous spectra, i.e., their eigenvalues are $\{-\infty < \alpha_1 < \infty\}$ and $\{-\infty < \alpha_2 < \infty\}$, respectively. Because \hat{a}_1 and \hat{a}_2 are observables, the appropriate orthonormality and completeness conditions for their eigenkets are therefore,

$${}_1\langle\alpha'_1|\alpha_1\rangle_1 = \delta(\alpha_1 - \alpha'_1) \quad \text{and} \quad {}_2\langle\alpha'_2|\alpha_2\rangle_2 = \delta(\alpha_2 - \alpha'_2),$$

$$\hat{I} = \int_{-\infty}^{\infty} d\alpha_1 |\alpha_1\rangle_1 {}_1\langle\alpha_1| = \int_{-\infty}^{\infty} d\alpha_2 |\alpha_2\rangle_2 {}_2\langle\alpha_2|.$$

(a) Use the Heisenberg uncertainty principle to show that $|\alpha_1\rangle_1$ and $|\alpha_2\rangle_2$ have infinite average energy, i.e., that $\langle\hat{H}\rangle = \hbar\omega(\langle\hat{a}_1^2\rangle + \langle\hat{a}_2^2\rangle) = \infty$ for these states.

- (b) We want to determine the relationship between the eigenkets $|\alpha_1\rangle_1$ and $|\alpha_2\rangle_2$. Use the results of Problem 4.4 to show that

$${}_2\langle\alpha_2|\alpha_1\rangle_1 = \exp(-2j\alpha_1\alpha_2){}_2\langle 0|0\rangle_1.$$

Hint: The power series expansion of $\hat{A}_1(\xi)$ can be used to show that $|\alpha_2\rangle_2$ is an eigenket of this translation operator; likewise $|\alpha_1\rangle_1$ is an eigenket of the translation operator $\hat{A}_2(\xi)$.

- (c) Find $|\!|_2\langle 0|0\rangle_1|^2$ by evaluating

$${}_2\langle\alpha'_2|\alpha_2\rangle_2 = {}_2\langle\alpha'_2|\hat{I}|\alpha_2\rangle_2 = {}_2\langle\alpha'_2|\left(\int_{-\infty}^{\infty} d\alpha_1 |\alpha_1\rangle_1\langle\alpha_1|\right)|\alpha_2\rangle_2,$$

using the result of (b). Assume that ${}_2\langle 0|0\rangle_1$ is positive real to completely pin down ${}_2\langle\alpha_2|\alpha_1\rangle_1$.