

# Dissipative ‘Groups’ and the Bloch Ball

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**Abstract.** We show that a quantum control procedure on a two-level system including dissipation gives rise to the semi-group corresponding to the Lie algebra  $gl(3, R) \oplus R^3$ . The physical evolution may be modelled by the action of this semi-group on a 3-vector as it moves inside the Bloch sphere, in the Bloch ball.

## 1. Introduction

Recent developments in quantum computing have emphasized the need for a realistic analysis of dissipation in systems which have the potential for use as qubits. In this note we discuss the effects of control and dissipation on a two-level system. For a single qubit *pure* state, it is well known that the unitary evolution may be visualised as the movement of a vector, the Bloch vector, on the surface of a 2-sphere, the Bloch sphere. In this note we extend the idea to a two-level *mixed* state. For this system, unitary evolution is on a spherical shell within the Bloch Sphere. Dissipation causes more general motion within the Bloch ball. This motion corresponds to the action of a certain *semi-group*.

We also show that the effects of dissipation may not be compensated by the interaction with the external control. However, taking into account the effects of measurement, which may be modelled by certain projection operators, the dissipative effects may indeed be modified, allowing more effective control of the system.

## 2. Dynamics of dissipative quantum control systems

In pure-state quantum mechanics the state of the system is usually represented by a wavefunction  $|\Psi\rangle$ , which is an element of a Hilbert space  $\mathcal{H}$ . For dissipative quantum systems, however, a quantum statistical mechanics formulation is necessary since dissipative effects can and do convert pure states into statistical ensembles and vice versa. In this case, the state of the system is represented by a density operator  $\hat{\rho}$ , whose diagonal elements determine the populations of the energy eigenstates, while the off-diagonal elements determine the coherences between energy eigenstates, which distinguish coherent superposition states  $|\Psi\rangle = \sum_{n=1}^N c_n |n\rangle$  from statistical ensembles of energy eigenstates (i.e., mixed states)  $\hat{\rho} = \sum_{n=1}^N w_n |n\rangle\langle n|$ . For a non-dissipative system the time evolution of the density matrix  $\hat{\rho}(t)$  with  $\hat{\rho}(t_0) = \hat{\rho}_0$  is governed by

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}_0\hat{U}(t)^\dagger, \quad (1)$$

where  $\hat{U}(t)$  is the time-evolution operator satisfying the Schrodinger equation

$$i\hbar \frac{d}{dt} \hat{U}(t) = \hat{H}(\mathbf{f})\hat{U}(t), \quad \hat{U}(0) = \hat{I}, \quad (2)$$

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where  $\hat{I}$  is the identity operator.  $\hat{\rho}(t)$  also satisfies the quantum Liouville equation

$$i\hbar \frac{d}{dt} \hat{\rho}(t) = [\hat{H}(\mathbf{f}), \hat{\rho}(t)] = \hat{H}(\mathbf{f})\hat{\rho}(t) - \hat{\rho}(t)\hat{H}(\mathbf{f}). \quad (3)$$

$\hat{H}(\mathbf{f})$  is the total Hamiltonian of the system, which depends on a set of control fields  $f_m$ :

$$\hat{H}(\mathbf{f}) = \hat{H}_0 + \sum_{m=1}^M f_m(t) \hat{H}_m, \quad (4)$$

where  $\hat{H}_0$  is the internal Hamiltonian and  $\hat{H}_m$  is the interaction Hamiltonian for the field  $f_m$  for  $1 \leq m \leq M$ .

The advantage of the Liouville equation (3) over the unitary evolution equation (1) is that it can easily be adapted to dissipative systems by adding a dissipation (super-)operator  $\mathbf{L}_D[\hat{\rho}(t)]$ :

$$i\hbar \dot{\rho}(t) = [\hat{H}_0, \hat{\rho}(t)] + \sum_{m=1}^M f_m(t) [\hat{H}_m, \hat{\rho}(t)] + i\hbar \mathbf{L}_D[\hat{\rho}(t)]. \quad (5)$$

In general, uncontrollable interactions of the system with its environment lead to two types of dissipation: phase decoherence (dephasing) and population relaxation (decay). The former occurs when the interaction with the environment destroys the phase correlations between states, which leads to a decay of the off-diagonal elements of the density matrix:

$$\dot{\rho}_{kn}(t) = -\frac{i}{\hbar} ([\hat{H}(\mathbf{f}), \hat{\rho}(t)])_{kn} - \Gamma_{kn} \rho_{kn}(t) \quad (6)$$

where  $\Gamma_{kn}$  (for  $k \neq n$ ) is the dephasing rate between  $|k\rangle$  and  $|n\rangle$ . The latter happens, for instance, when a quantum particle in state  $|n\rangle$  spontaneously emits a photon and decays to another quantum state  $|k\rangle$ , which changes the populations according to

$$\dot{\rho}_{nn}(t) = -\frac{i}{\hbar} ([\hat{H}(\mathbf{f}), \hat{\rho}(t)])_{nn} + \sum_{k \neq n} [\gamma_{nk} \rho_{kk}(t) - \gamma_{kn} \rho_{nn}(t)] \quad (7)$$

where  $\gamma_{kn} \rho_{nn}$  is the population loss for level  $|n\rangle$  due to transitions  $|n\rangle \rightarrow |k\rangle$ , and  $\gamma_{nk} \rho_{kk}$  is the population gain caused by transitions  $|k\rangle \rightarrow |n\rangle$ . The population relaxation rate  $\gamma_{kn}$  is determined by the lifetime of the state  $|n\rangle$ , and for multiple decay pathways, the relative probability for the transition  $|n\rangle \rightarrow |k\rangle$ . Phase decoherence and population relaxation lead to a dissipation superoperator (represented by an  $N^2 \times N^2$  matrix) whose non-zero elements are

$$\begin{aligned} (\mathbf{L}_D)_{kn, kn} &= -\Gamma_{kn} & k \neq n \\ (\mathbf{L}_D)_{nn, kk} &= +\gamma_{nk} & k \neq n \\ (\mathbf{L}_D)_{nn, nn} &= -\sum_{n \neq k} \gamma_{kn}. \end{aligned} \quad (8)$$

Population decay and dephasing allow us to overcome kinematical constraints such as unitary evolution to create statistical ensembles from pure states, and pure states from statistical ensembles, which is important for many applications such as optical pumping. However, there are instances when this is not desirable such as in quantum computing, where these effects destroy quantum information. Thus, there are situations when we would like to prevent decay and dephasing. A cursory glance at the quantum Liouville equation for coherently driven, dissipative systems (5) suggests that it might be possible to prevent population and phase relaxation by applying suitable control fields such that

$$\sum_{m=1}^M f_m(t) [\hat{H}_m, \hat{\rho}(t)] + i\hbar \mathbf{L}_D[\hat{\rho}(t)] = 0. \quad (9)$$

Unfortunately, however, a more careful analysis reveals that this is *not* possible, in general, as we shall now show explicitly for a two-level system, or qubit in quantum computing parlance.

### 3. Dynamics of a 2-level system subject to control, decay and dephasing

The Hamiltonian for a driven two-level system with energy levels  $E_1 < E_2$  is

$$\hat{H}[\mathbf{f}(t)] = \hat{H}_0 + f_1(t)\hat{H}_1 + f_2(t)\hat{H}_2 \quad (10)$$

where  $\hat{H}_0$  is the internal Hamiltonian and  $\hat{H}_1$  and  $\hat{H}_2$  represent interaction Hamiltonians with independent (real-valued) control fields  $f_1(t)$  and  $f_2(t)$ ,

$$\hat{H}_0 = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \quad \hat{H}_1 = d_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{H}_2 = d_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

$d_1, d_2$  are the (real-valued) dipole moments for the transition and  $\omega = (E_2 - E_1)/\hbar$  is the transition frequency.

We can re-write the Liouville equation in matrix form in a higher dimensional space, often referred to as Liouville space. Straightforward computation shows that

$$\frac{d}{dt}|\rho(t)\rangle\rangle = \mathbf{L}|\rho(t)\rangle\rangle = [(1/i\hbar)(\mathbf{L}_0 + f_1(t)\mathbf{L}_1 + f_2(t)\mathbf{L}_2) + \mathbf{L}_D]|\rho(t)\rangle\rangle \quad (11)$$

where  $|\rho(t)\rangle\rangle = (\rho_{11}(t), \rho_{12}(t), \rho_{21}(t), \rho_{22}(t))^T$

$$\mathbf{L}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\hbar\omega & 0 & 0 \\ 0 & 0 & +\hbar\omega & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{L}_1 = d_1 \begin{pmatrix} 0 & -1 & +1 & 0 \\ -1 & 0 & 0 & +1 \\ +1 & 0 & 0 & -1 \\ 0 & +1 & -1 & 0 \end{pmatrix}$$

$$\mathbf{L}_2 = d_2 \begin{pmatrix} 0 & -i & -i & 0 \\ +i & 0 & 0 & -i \\ +i & 0 & 0 & -i \\ 0 & +i & +i & 0 \end{pmatrix} \quad \mathbf{L}_D = \begin{pmatrix} -\gamma_{21} & 0 & 0 & \gamma_{12} \\ 0 & -\Gamma & 0 & 0 \\ 0 & 0 & -\Gamma & 0 \\ \gamma_{21} & 0 & 0 & -\gamma_{12} \end{pmatrix}$$

$\gamma_{12}$  is the rate of population relaxation from  $|2\rangle$  to  $|1\rangle$ ,  $\gamma_{21}$  is the rate of population relaxation from  $|1\rangle$  to  $|2\rangle$  (usually zero), and  $\Gamma$  is the dephasing rate.

Notice that the matrix elements of the Liouville operators  $\mathbf{L}_1$  and  $\mathbf{L}_2$  are zero where the matrix elements of the dissipation operator  $\mathbf{L}_D$  are non-zero, and vice versa. Thus, no matter how we choose the control fields, we cannot cancel the effect of the dissipative terms. The best we can do is to use coherent control to implement quantum error correction schemes to restore decayed/dephased quantum states to their correct values. An early contribution to the theory of continuous feedback for such systems is contained in [1], while a more recent scheme of continuous quantum error correction involving weak measurements and feedback has been proposed in [2].

### 4. Dynamical Semi-group and the Bloch Ball

The more usual real vector form for  $\rho$  is  $\rho_B \equiv (\rho_{1,2} + \rho_{2,1}, i(\rho_{1,2} - \rho_{2,1}), \rho_{1,1} - \rho_{2,2}, \rho_{1,1} + \rho_{2,2})$ . In the general case this is referred to as the *coherence vector*[3]. Since for the systems under consideration we shall take  $\rho_{1,1} + \rho_{2,2}$  as constant (no population loss) only the first three components of  $\rho_B$  transform under the dynamics. For *pure states* the norm is constant - thus generating motion on the surface of a 2-sphere, the Bloch sphere. For our more general scenario, motion takes place in the interior of this sphere, the *Bloch ball*.

The family of control hamiltonians generate the Lie algebra  $u(2)$ , in this case a completely controllable system[4]. The Lie algebra generated by the matrices corresponding

to Eq.(3) acting on  $\rho_B$  is the inhomogeneous algebra  $gl(3, R) \oplus R^3$ . The evolution of the system in time is determined by  $\exp(\mathbb{L}t)$ . Noting that the eigenvalues of  $\mathbb{L}_D$  are (always) negative, the demand that our set of operators remain bounded gives in effect a *semi-group*, with only unlimited *positive* values of  $t$  permitted. These considerations may be generalized to any dimensions [5].

## 5. Conclusions

In this note we showed how the effects of control and dissipation can be treated geometrically, by the movement of the coherence vector inside the Bloch ball. We incidentally noted that dissipative effects could not be compensated by control dynamics alone. Without measurements and feedback, dissipation usually forces the system into an equilibrium state. One can see that without any control this state corresponds to a point on the z-axis of the Bloch sphere; with constant controls one can show that the state converges to a point on an ellipse inside the Bloch ball. The coordinates of these attractors can easily be expressed as a function of the dissipative terms.

Intuitively, error correction is not possible with control fields alone because the dissipative terms tend to pull us inside the Bloch ball and, without quantum measurement feedback, we can't get back to the surface of the ball because the control fields can only perform rotations.

The analysis can be treated by traditional Lie algebraic methods, by use of a dynamical Lie algebra. In the specific case of a qubit treated here, this algebra is the inhomogeneous semi-direct sum  $gl(3, R) \oplus R^3$ . The control fields generate a rotation algebra within this larger algebra. Imposing boundedness conditions on the evolution of the dynamics obtained by exponentiation of this algebra leads to a semi-group description of the evolution.

Although due to restrictions of space we have treated explicitly only the case of a two-level system, the methods can be generalised without difficulty to any finite level system. For an  $N$ -level system, the coherence vector is essentially an  $N^2 - 1$  real component vector (for population-preserving dynamics) whose motion is restricted to the  $N^2 - 2$ -dimensional Bloch sphere for non-dissipative systems, and to the interior of the corresponding Bloch ball in general. The associated inhomogeneous real algebra is given by semi-direct sum  $gl(N^2 - 1, R) \oplus R^{(N^2-1)}$ , and this gives rise by means of exponentiation to the corresponding semi-group which determines the dynamics. Finally, for *quasi-spin* systems, defined by symmetric population decay rates, the inhomogeneous (translation-like) terms disappear.

## References

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