

Solutions to Homework Assignment No. 4

College Algebra (math 111)

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EXERCISES TO SECTION 2.6:

Exercises 1-36: Find all real solutions of the following equations and check result graphically.

1. $x^3 - 2x^2 - 3x = 0$. Factoring out x and completing the square of the remaining quadratic factor yields $x(x^2 - 2x - 3) = x[(x - 1)^2 - 4] = 0$. Thus we have the solutions $x_2 = 0$ and $x_{1,3} = 1 \pm \sqrt{4}$ i.e. $x_1 = -1$ and $x_3 = 3$.
4. $20y^3 - 125y = 0$. Factoring out $5y$ yields $5y(4y^2 - 25) = 0$. And noting that $(2y + 5)(2y - 5) = 4y^2 - 25$ leads to $5y[(2y + 5)(2y - 5)] = 0$. Thus we have the solutions $y_1 = -\frac{5}{2}$, $y_2 = 0$ and $y_3 = \frac{5}{2}$.
5. $x^3 - 3x^2 - x + 3 = 0$. Factoring out “intelligently” yields

$$x^2(x - 3) - (x - 3) = [x^2 - 1](x - 3) = [(x + 1)(x - 1)](x - 3) = 0.$$

Thus the solutions are $x_1 = -1$, $x_2 = 1$, $x_3 = 3$.

8. $36t^4 - 29t^2 - 7 = 0$. Substitution of $u = 6t^2$ lead to a quadratic equation in u which we can solve by completing the square:

$$u^2 - \frac{29}{6}u + 9 = (u - \frac{29}{12})^2 - \frac{29^2}{12^2} + 9 = 0$$

This yields

$$u_{1,2} = \frac{29}{12} \pm \sqrt{\frac{29^2 - 9 \cdot 144}{12^2}}$$

i.e. there is no real solution because $29^2 - 9 \cdot 144 = -455 < 0$ i.e. the discriminant is negative.

9. $\frac{1}{t^2} + \frac{8}{t} + 15 = 0$. We multiply the equation by t^2 to obtain a quadratic equation $1 + 8t + 15t^2 = 0$ which we can solve by completing the square

$$t^2 + \frac{8}{15}t + \frac{1}{15} = (t + \frac{4}{15})^2 - \frac{16}{15^2} + \frac{1}{15} = (t + \frac{4}{15})^2 - \frac{1}{15^2} = 0.$$

Thus we have the solutions $t_{1,2} = -\frac{4}{15} \pm \frac{1}{15}$ i.e. $t_1 = -\frac{1}{3}$ and $t_2 = -\frac{1}{5}$. Since the original equation is defined for both values we really have a solution.

Be careful! Multiplication of an equation with a variable factor is not necessarily an equivalence transformation. Note the solution of the fraction-free equation might be a zero of the denominator of a fraction of the original equation and thus the original equation could be undefined. We have to exclude such “solutions”.

12. $9t^{2/3} + 24t^{1/3} + 16 = 0$. Substitution of $u = 3t^{1/3}$ results $u^2 + 8u + 16 = (u + 4)^2 = 0$. Thus $u = -4$ and $t = (u/3)^3 = -64/27$.

13. $\sqrt{2x} - 10 = 0$. Separate the square root $\sqrt{2x} = 10$ and square both sides of the equation to get $2x = 100$ which yields $x = 50$. Check that $x = 50$ really solves the original equation.

Since taking the square of an equation is not an equivalence transformation of the equation, we always have to check if the solution of the “squared equation” is also a solution of the original one. Every solution of the original equation solves the “squared equation” but not vice versa!

17. $\sqrt{x+1} - 3x = 1$. Isolate the square root $\sqrt{x+1} = 3x + 1$. Now take the square of both sides of the equation $x + 1 = (3x + 1)^2$. This is a quadratic equation that can be re-written as $9x^2 + 5x = x(9x + 5) = 0$. Thus the solutions of the square of the equation are $x = 0$ and $x = -\frac{5}{9}$. Verify that $\sqrt{0+1} - 3 \cdot 0 = 1$ but

$$\sqrt{-\frac{5}{9} + 1} - 3\left(-\frac{5}{9}\right) = \sqrt{\frac{4}{9}} + \frac{5}{3} = \frac{7}{3} \neq 1$$

i.e. only $x = 0$ is a solution of the original equation!

18. $\sqrt{x+5} = \sqrt{x-5}$. Taking the square of both sides leads to $x+5 = x-5$ i.e. an equation which has no solution. Thus the original equation doesn't have a solution either.
21. $\sqrt{x+5} + \sqrt{x-5} = 10$. Taking the square of both sides we obtain

$$x+5+2\sqrt{x+5}\sqrt{x-5}+x-5 = 100 \Leftrightarrow 2x+2\sqrt{(x+5)(x-5)} = 100 \Leftrightarrow \sqrt{(x+5)(x-5)} = 50-x.$$

Taking the square of both sides of the last equation yields $(x+5)(x-5) = (50-x)^2$ or $x^2-25 = x^2-100x+50^2$ i.e. the quadratic terms cancel out and we are left with a linear equation

$$x = \frac{50^2 + 25}{100} = \frac{5^2 \cdot 10^2 + 5^2}{10^2} = \frac{101}{4}.$$

Don't forget to verify the result:

$$\sqrt{\frac{101+20}{4}} + \sqrt{\frac{101-20}{4}} = \frac{1}{2}\sqrt{121} + \frac{1}{2}\sqrt{81} = \frac{1}{2}(11+9) = 10$$

i.e. we really have a solution of the original equation.

25. $3x(x-1)^{1/2} + 2(x-1)^{3/2} = 0$ can be factored as $\sqrt{x-1}[3x+2(x-1)] = \sqrt{x-1}[5x-2] = 0$. Thus we have the solutions $x_1 = \frac{2}{5}$ and $x_2 = 1$.
29. $\frac{1}{x} - \frac{1}{x+1} = 3$. We multiply both sides by $x(x+1)$ to obtain a fraction-free equation $x-1-x = 3x(x-1)$ which rewrites as $9x^2 - 3x + 1 = (3x - \frac{1}{2})^2 + \frac{3}{4} = 0$ — an equation which obviously can not have a real solution.
33. $|2x-1| = 5$. First thing to do with an absolute value equation is split it up to get rid of the absolute values:

$$|2x-1| = 5 \Leftrightarrow \begin{cases} 2x-1 = 5 & 2x-1 \geq 0 \\ 1-2x = 5 & x < 0.5 \end{cases} \Leftrightarrow x \geq 0.5$$

The first linear equation has the solution $x = 3 > 0.5$ and thus this is really a solution of the original equation. The second linear equation has the solution $x = -2 < 0.5$ i.e. it is also a solution.

Exercises 37-56: Approximate the real solutions using a graphing utility.

37. $x^4 - x^3 + x - 1 = 0$. We find the approximate solutions $(-1, -9E-14)$ and $(1, 0)$. It is easy to verify that $x = \pm 1$ are indeed the exact solutions.
41. $2x + 9\sqrt{x} - 5 = 0$. The approximate solution is $(0.25, -1E-13)$. We can verify that $x = \frac{1}{4}$ is indeed the exact solution.
44. $x + \sqrt{31-9x} - 5 = 0$. The approximate solutions are $(-2, 0)$ and $(3, 0)$. In fact, these are also the exact solutions.

These few examples demonstrate how useful a graphing utility can be! In all cases we found all exact solutions by approximating the solution and verifying that the result was indeed exact.

Some text problems:

Exercise 58: Say if three students share the apartment everybody pays x bucks a month i.e. the total monthly rent is $3x$. Now we now that if we add a 4th student then the monthly rent for each person will be only $x - 75$. Thus assuming of course that the rent for the apartment does not depend on the number of occupants we have $3x = 4(x - 75)$ which is a simple linear equation which the solution $x = 300$. Hence, the monthly rent for the apartment is \$900.

Exercise 60: We know that on the way to their vacation lodge the family drove a total distance of 1080 miles in t hours and their average speed was x . Assuming they took the same way home we must have $1080 = xt$ and $1080 = (x-6)(t+2.5)$. The first equation yields e.g. $t = \frac{1080}{x}$ and this substituted in the second equation results

$$1080 = (x-6)\left(\frac{1080}{x} + 2.5\right) = 1080 + 2.5x - \frac{6 \cdot 1080}{x} - 15 \Leftrightarrow 2.5x - \frac{6 \cdot 1080}{x} - 15 = 0.$$

Now multiplication by x yields the quadratic equation

$$2.5x^2 - 15x - 6480 = 0 \Leftrightarrow x^2 - 6x - 2592 = 0 \Leftrightarrow (x-3)^2 = 2592.$$

This yields $x = 3 + \sqrt{2592} \approx 54$ i.e. on their way to the lodge their average speed was about 54 miles per hour.

Exercise 62: Assume we deposit x dollar and the annual percentage rate is p . Then after 1 year we have $(1+p)x$ dollar. After 2 years we have $(1+p)(1+p)x = (1+p)^2x$ dollar and so forth. Thus after n years we'd have $(1+p)^n x$ dollar. Now let $x = 10000$, $n = 10$ and we know after 10 year we have \$25000. Then the equation is

$$25000 = 10000(1+p)^{10} \Leftrightarrow 2.5 = (1+p)^{10} \Leftrightarrow p = \sqrt[10]{2.5} - 1 \approx 0.096$$

If we got 35000 instead of 25000 the annual interest rate was $\sqrt[10]{3.5} - 1 \approx 0.133$ or 13.3 percent.

EXERCISES TO SECTION 2.7: LINEAR INEQUALITIES

Exercises 1-4: Determine if given values are Solutions to an Inequality

One way to do this is simply plug in the given value(s) and simplify algebraically and see if in the end a true statement is the result.

- $5x - 12 > 0$. For $x = 3$ we have $15 - 12 = 3 > 0$. $x = -3$ yields $-15 - 12 < 0$. $x = \frac{5}{2}$ leads to $12.5 - 12 > 0$ and $x = \frac{3}{2}$ results $7.5 - 12 < 0$. Thus, $x = 3$ and $x = 5/2$ satisfy the inequality while the other two values don't.
- $-1 < \frac{3-x}{2} \leq 1$. $x = 0$: $-1 < \frac{3}{2} \leq 1$. Certainly $-1 < \frac{3}{2}$ but $\frac{3}{2} > 1$. Thus 0 is not a solution to the inequality. $x = 1$: $-1 < \frac{3-1}{2} = 1 \leq 1$ i.e. $x = 1$ is a solution. $x = \sqrt{5}$: Note that $4 < 5 < 9$ and thus $2 = \sqrt{4} < \sqrt{5} < \sqrt{9} = 3$ and thus $0 < 3 - \sqrt{5} < 1$ and thus $-1 < 0 < \frac{3-\sqrt{5}}{2} < \frac{1}{2} \leq 1$ i.e. $\sqrt{5}$ is a solution. $x = 5$: $\frac{3-5}{2} = -1$ i.e. 5 is no solution.

Exercises 5-12: Match Inequality with its Graph

We have 5c, 6h, 7f, 8e, 9g, 10a, 11b, 12d.

Exercises 13-38: Solve Inequality Algebraically

Please don't forget:

You can add or subtract positive or negative or values from both sides of an inequality or if it is a double inequality from all three parts simultaneously without changing the inequality. You can also multiply or divide all parts by positive factors. However, if you multiply an inequality by a negative factor you have to reverse all inequality signs i.e. $<$ becomes $>$ and $>$ becomes $<$ and \leq becomes \geq and \geq becomes \leq .

- $4x < 12$ is equivalent to $x < \frac{12}{4} = 3$.
- $x - 5 \geq 7$ is equivalent to $x \geq 7 + 5 = 12$.
- $1 < 2x + 3 < 9$ is equivalent to $1 - 3 < 2x < 9 - 3$ or $4 < 2x < 6$ and that is equivalent to $2 < x < 3$.
- $0 \leq \frac{x+3}{2} < 5$ is equivalent to $0 \cdot 2 \leq x + 3 < 5 \cdot 2 = 10$ is equivalent to $0 - 3 \leq x < 10 - 3$ i.e. $0 \leq x < 7$.
- $\frac{3}{4} > x + 1 > \frac{1}{4}$ is equivalent to $\frac{3}{4} - 1 > x > \frac{1}{4} - 1$ or $-\frac{1}{4} > x > -\frac{3}{4}$.
- $|2x| < 6$ is equivalent to $-6 < 2x < 6$ is equivalent to $-3 < x < 3$.
- $|x/2| > 3$ is equivalent to $x/2 > 3$ or $x/2 < -3$ and that is equivalent to $x > 6$ or $x < -6$.
- $|5x| > 10$ is equivalent to $5x > 10$ or $5x < -10$ and that is equivalent to $x > \frac{10}{5} = 2$ or $x < -2$.
- $|x - 20| \geq 4$ is equivalent to $x - 20 \geq 4$ or $20 - x \geq 4$. $x - 20 \geq 4$ is equivalent to $x \geq 24$. But beware $20 - x \geq 4$ is equivalent to $20 \geq 4 + x$ is equivalent to $20 - 4 \geq x$ i.e. $16 \geq x$. Thus, both inequalities together yield $x \leq 16$ or $x \geq 24$.
- $|x - 5| < 0$. Since the absolute value function is non-negative, this inequality has no solution.
- $|x - 5| \geq 0$. Since $|\cdot|$ is always ≥ 0 this inequality is true for all $x \in \mathbb{R}$.

EXERCISES TO SECTION 2.8:

- We can re-write the inequality in the form

$$\frac{-4x + 25}{5} - \frac{3x - 10}{10} \geq 0.$$

Now we use our graphing calculator to approximate the zeros of this expression and to determine where it is non-negative. I get the root $x_0 = 5.4545455$ and the expression above is positive for $x < x_0$.

44. $-1 \leq 13 - 2x \leq 9$. One way we do it is define the lines $y_1 = 13 - 2x$, $y_2 = 9$ and $y_3 = -1$ and use our graphing tool to find the intersections between y_1 and y_2 as well as y_1 and y_3 using the zoom, trace and calc function. I obtain that y_1 intersects y_2 at $x = 2$ and y_3 at $x = 7$. Thus $2 \leq x \leq 7$.
45. We plot the expression $|x - 3|/2 - 5$ and find its roots approximately and determine the interval where it is positive. I get $-7 \leq x \leq 13$.

Exercises 49-52: Find the intervals on the real line where the **radicand is non-negative**.

49. We have

$$3 - x \geq 0 \quad \Leftrightarrow \quad 3 - x + x \geq x \quad \Leftrightarrow \quad 3 \geq x$$

i.e. $x \leq 3$. Be careful. If you subtract 3 from both sides you will get $3 - x - 3 \geq -3$ i.e. $-x \geq 3$. Now you have to multiply by (-1) and that reverses the \geq to a \leq i.e. $x \geq 3$.

50. $x - 10 \geq 0$ for $x \geq 10$.
51. $7 - 2x \geq 0$ for $x \leq 3.5$.
52. $6x + 15 \geq 0$ for $x \geq -\frac{15}{6}$.

Exercises 53-60: Absolute Value Notation

53. $|x| \leq 2$
54. $|x| > 2$
55. Notice that the absolute value function has a symmetry property i.e. if you want to express $x \leq 6$ or $x \geq 12$ in absolute value notation then you have to find the center of the interval (6, 12) first. Obviously, this is 9. Thus, $\{x : x \leq 6 \text{ or } x \geq 12\}$ is the set of all points x which are at least 3 units from the center 9 apart. This translates now into absolute value notation as $|x - 9| \geq 3$.
56. Similarly, to express the set $\{x : -6 \leq x \leq 2\}$ in absolute value notation you have to find the center of the interval $[-6, 2]$. This is simply $(-6 + 2)/2 = -2$. You want all points x that are at most $2 - (-2) = 4$ units from the center point -2 apart. That translates into absolute value notation as follows: $|x - (-2)| \leq 4$ i.e. $|x + 2| \leq 4$.
57. All real numbers within 10 units of 12 are $\{x : |x - 12| \leq 10\}$.
58. All real numbers at least 5 units from 8 are $\{x : |x - 8| \geq 5\}$.
59. All real numbers whose distances from -3 are more than 5 are $\{x : |x - (-3)| > 5\}$.
60. All real numbers whose distances from -6 are no more than 7 are $\{x : |x - (-6)| \leq 7\}$.

Note: the verbal expression "no more" translates into \leq , "at least" translates to \geq , "more than" translates to $>$ and less than becomes $<$.

Some Text problems:

Exercise 62: The cost for making n copies on your own copier are $C_1(n) = 3000 + 0.03n$. The cost for making n copies in the photocopy center are $C_2(n) = 0.1n$. Now, let's assume the purchase of a copier is justified if the total costs $C_1(n)$ are no more than if you made the copies in the photocopy center i.e.

$$C_1(n) \leq C_2(n) \quad \Leftrightarrow \quad 3000 + 0.03n \leq 0.1n \quad \Leftrightarrow \quad 3000 \leq 0.07n \quad \Leftrightarrow \quad n \geq 42858.$$

Thus, to justify your own copier you'd have to make at least 42858 copies in 4 years.

Exercise 64: We invest $x = 750$ dollar at a rate an interest rate of r and we assume simple interest i.e. the output after t years is $x(1 + rt)$. If we want to have at least 1050 bucks after two years this means we want

$$750(1 + 2r) \geq 1050 \quad \Leftrightarrow \quad 1 + 2r \geq \frac{1050}{750} \quad \Leftrightarrow \quad 2r \geq \frac{1050}{750} - 1 = \frac{300}{750} \quad \Leftrightarrow \quad r \geq \frac{150}{750} = \frac{1}{5}.$$

i.e. we'd have to invest our money at an interest rate of 20%.

Exercises 65/66. Cost & Revenue