Problem Set 3

Problem 1: Adiabatic Control.

(a) State the adiabatic theorem for controlled dynamical systems and explain how it can be used for control by adiabatic passage.

(b) The Hamiltonian for a three-level Λ system (see lecture slides) simultaneously driven by two independent control fields f_1 and f_2, driving the $1,2$ and $2,3$ transitions, respectively, is

$$\hat{H}[\mathbf{f}(t)] = \begin{pmatrix} \epsilon_1 & d_1 f_1(t) & 0 \\ d_1 f_1(t) & \epsilon_2 & d_2 f_2(t) \\ 0 & d_2 f_2(t) & \epsilon_3 \end{pmatrix}$$

where ϵ_k are the energy levels and d_k the dipole moments of the system. Show that if we apply fields of the form $f_k(t) = A_k(t)\cos(\omega_k t + \phi_k)$ with $\omega_k = \epsilon_{k+1} - \epsilon_k$ for $k = 1,2$, then transforming to a rotating frame and making the rotating wave approximation leads to the RWA Hamiltonian

$$\hat{H}[\Omega_k(t)] = \begin{pmatrix} 0 & \Omega_1(t) & 0 \\ \Omega_1(t) & 0 & \Omega_2(t) \\ 0 & \Omega_2(t) & 0 \end{pmatrix}$$

where $\Omega_k(t) = A_k(t) d_k / 2\hbar$. (When is the RWA valid?)

(c) Show that the eigendecomposition of the RWA Hamiltonian is

$$\begin{aligned} \lambda_0 &= 0, & |\Psi_0(t)\rangle &= \cos\theta(t)|1\rangle - \sin\theta(t)|3\rangle \\ \lambda_\pm &= \pm\Omega, & |\Psi_\pm(t)\rangle &= \Omega_1|1\rangle \pm \Omega|2\rangle + \Omega_2|3\rangle \end{aligned}$$

where $\Omega = \sqrt{\Omega_1^2 + \Omega_2^2}$ and $\theta(t) = \arctan[\Omega_1(t)/\Omega_2(t)]$.

(d) Use the eigendecomposition of the RWA Hamiltonian to explain how we can adiabatically transfer population from state $|1\rangle$ to $|3\rangle$ without populating state $|2\rangle$, and describe how we could realize this scheme experimentally.

(e) Extra fun: Write a program that allows you to enter two overlapping Gaussian control pulses and solve the resulting Schrodinger equation for the pure input state $|1\rangle$.
Experimentally determine the required field strengths and pulse separation/overlap to ensure that the system undergoes adiabatic passage.
Parametrize the fields and optimize the parameters to ensure rapid adiabatic passage with maximal suppression of dynamic excitation.
What happens when you apply the STIRAP pulse sequence to a different initial state, i.e., $|2\rangle$, $|3\rangle$ or a superposition state?

Problem 2: Optimal Control.

(a) Write down a variational functional $\mathcal{J}(\mathbf{f}(t), A_v, \rho_v)$ for the problem of maximizing an observable represented by the Hermitian operator $\hat{A}$ for a system satisfying the quantum Liouville equation $\dot{\rho}(t) = -i[\hat{H}, \hat{\rho}] + \mathcal{L}_D[\hat{\rho}]$ with $\hat{H} = \hat{H}[\mathbf{f}(t)] = \hat{H}_0 + \sum_{m=1}^{M} f_m(t)\hat{H}_m$. Explain the meaning of the terms and variables.

(b) A necessary condition for $\mathcal{J}$ to have an extremum is that the independent variations of $\delta J/\delta A_v$, $\delta J/\delta \rho_v$, and $\delta J/\delta f_m$ vanish. Use this condition to derive the Euler-Lagrange equations. (NB: There is a misprint in the lecture notes. Can you spot it?)

(c) Describe an iterative algorithm to solve the Euler-Lagrange equations.

(d) What can you say about the convergence properties of the algorithm? [Ambitious types: Could you prove the results?]

(e) How would you implement the resulting complicated shaped control pulses experimentally using standard laser pulse shaping equipment?

(f) Extra fun: Implement the algorithm described above and experiment with the various parameters.

Problem 3: Direct Laboratory Optimization.

(a) How does direct laboratory optimization work? How does it differ from model-based open-loop control design? What are the advantages/disadvantages?

(b) Describe briefly two algorithms for direct laboratory optimization (learning control).

(c) Extra fun: Simulate a direct laboratory optimization experiment on your computer. (Hint: you need to implement two core routines: A measurement simulator and your chosen optimization routine.)

Problem 4: Stochastic feedback control I.

(a) Describe a classical feedback control scheme to track the orbit of a target state under the free evolution Hamiltonian $\hat{H}_0$ for a bilinear Hamiltonian system using Lyapunov functions. What are the problems with applying this type of feedback control to quantum systems?

(b) Explain how homodyne detection works for a coherently driven two-level atom subject to spontaneous emission.

(c) Write down a stochastic differential equation for a coherently driven two-level atom subject to spontaneous emission (i) if there is no measurement or feedback, (ii) homodyne dectection but no feedback and (iii) homodyne detection with feedback. [1]

(d) Use the stochastic state equation to derive a stochastic master equation for the conditional density operator $\hat{\rho}_c$, and show that averaging over many quantum trajectories leads to a non-stochastic master equation of Lindblad form (see slide No. 9 on closed-loop control).

(e) Extra fun: Write a program to simulate stochastic trajectories and verify that the average evolution (i.e., the evolution averaged over many quantum trajectories) does indeed satisfy the non-stochastic master equation.

Problem 5: Stochastic feedback control II.

(a) Write down the interaction Hamiltonian for a cavity with a quantized field, and show that the RWA approximation leads to $\hat{H}_I(t) = i\sqrt{\gamma}[\hat{a}(t)\hat{b}^\dagger(t) - \hat{a}^\dagger(t)\hat{b}(t)]$, where $\hat{a}^\dagger$ is the cavity creation operator, $\hat{b}^\dagger$ is the creation operator for the external field, and γ is the cavity-field coupling strength.

(b) Define a stochastic input operator $\hat{b}_{in}$ and use it to show that, assuming Markovian dynamics, the stochastic evolution operator is

$$\hat{U}_I(t+dt,t) = \exp\left[\sqrt{\gamma}[\hat{a}(t)\hat{B}_{in}^\dagger(t) - \hat{a}^\dagger(t)\hat{B}_{in}(t)]\right].$$

(c) Use the 2nd order Taylor expansion of the evolution operator to show that an arbitrary cavity operator $\hat{s}(t)$ satisfies the explicit equation

$$\begin{aligned} d\hat{s}(t) &= \hat{U}(t+dt,t)^\dagger \hat{s}(t)\hat{U}(t+dt,t) - \hat{s}(t) \\ &= \sqrt{\gamma}[\hat{a}^\dagger \otimes d\hat{B}_{in} - \hat{a} \otimes d\hat{B}_{in}^\dagger] + \gamma\{(N-1)(\hat{a}^\dagger\hat{s}\hat{a} - \frac{1}{2}\hat{s}\hat{a}^\dagger\hat{a} - \frac{1}{2}\hat{a}^\dagger\hat{a}\hat{s}) \\ &\quad + N(2\hat{a}\hat{s}\hat{a}^\dagger - \hat{s}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{s}) + M[\hat{a}^\dagger,[\hat{a}^\dagger,\hat{s}]] + M^*[\hat{a},[\hat{a},\hat{s}]]\} \otimes \hat{1}_B \end{aligned} \tag{0.1}$$

where the time dependence of the operators on the RHS has been omitted for compactness. Show that for $\hat{s} = \hat{a}$ using the commutation relation $[\hat{a},\hat{a}^\dagger] = 1$ leads to the (implicit) equation for the cavity mode $\dot{\hat{a}}(t) = -\frac{\gamma}{2}\hat{a}(t) - \sqrt{\gamma}\hat{b}_{in}(t)$.

(d) Use the relation $d\hat{b}_{out}(t) = \hat{U}(t+dt,t)^\dagger\hat{b}_{in}(t)\hat{U}(t+dt,t)$ and the 2nd order Taylor expansion for $\hat{U}(t+dt,t)$ above to show that $\hat{b}_{out}(t) = \sqrt{\gamma}\hat{a}(t) + \hat{b}_{in}(t)$.

(e) Combine the two linear equations for the cavity mode and the input output relation, and use the Laplace transform to show that the gain or transfer function of the cavity is

$$\tilde{b}_{out}(s) = \frac{s-\gamma/2}{s+\gamma/2}\tilde{b}_{\rm in}(s)$$

(f) Write down input-output relations for feedback with a beamsplitter and the closed-loop transfer function.

(g) State Nyquist's stability condition and use it to show that the closed-loop system is asymptotically stable.

[1] You need not be able to derive these equations as this is non-trivial and was not explained in class, nor should you memorize them. I would *not* ask a question like this in closed book exam.